An exact solution for progressive capillary waves of arbitrary amplitude

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(*Received* 4 *March* 1957)

Summary

An exact solution is found in a fairly simple form for two-dimensional progressive waves of arbitrary amplitude on a fluid of unlimited depth, when only surface tension and not gravity is taken into account as the restoring force. The calculated wave forms are exhibited graphically for various amplitudes, and the relation between wave velocity and amplitude is plotted. The wave of greatest height occurs when the vertical distance between trough and crest is 0·730 wavelengths (compared with 0·142 for gravity waves). Higher waves are prevented from appearing by the enclosing of air bubbles in the troughs.

1. Introduction

Periodic gravity waves of finite amplitude have been investigated by Stokes (1847; 1880, pp. 197 & 314; see also Lamb 1932, § 250) and others, and a good approximate solution has been found. The gravity wave of greatest height has also been determined (Stokes 1880, p. 225). It is therefore interesting to see if analogous solutions can be obtained for purely capillary waves, for which gravity is neglected; it is particularly interesting as a first step towards investigating the problem of waves of finite amplitude under the combined effects of gravity and surface tension.

Capillary waves are found to be remarkable in that an exact solution exists for arbitrary amplitude. The case where the fluid has infinite depth was considered first, and, following Stokes, a series approach was made. Inspection revealed regularities in the coefficients, however, and the series obtained by assuming that these regularities continued in the higher terms was found to have a sum in closed form. This sum was then shown to be an exact solution. From this original investigation the present method, leading directly to the solution, was evolved. The present method indicates that there is also an exact solution if the fluid has finite uniform depth. The analysis is, however, rather complicated, involving elliptic functions, and the solution is not considered worth evaluating in detail.

The solution for infinite depth shows that the wave velocity given by the theory of waves of infinitesimal amplitude (Lamb 1932, § 265 & § 266) is accurate only for zero amplitude and that, if the wavelength is fixed, the velocity decreases as the amplitude increases. This behaviour is exactly opposite to that of gravity waves. The waves themselves are very rounded in section, and the wave of greatest height is reached when the surface bends back to touch itself, enclosing a bubble of air.

2. Equations of motion in the (ϕ, ψ)-plane

Our investigation will be confined to two-dimensional travelling waves on the surface of an ideal fluid which has infinite depth. It is convenient to choose Cartesian axes with x measured horizontally to the left and y vertically downwards. To make the flow steady we bring the waves to rest by superimposing a uniform velocity on the system, and therefore we assume that the undisturbed fluid (that is, the fluid at great depths) is moving in the positive x-direction with velocity c, the wave velocity.

If the motion is generated from rest, the flow must be irrotational, and so the equation of motion is

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0, \tag{1}$$

where ϕ is the velocity potential of the flow. The stream function ψ also satisfies this equation.

In a steady flow the surface is a streamline, say $\psi = 0$, and Bernoulli's equation holds, so we have

$$p/\rho+\tfrac{1}{2}q^2=\tfrac{1}{2}c^2. \tag{2}$$

Here p represents the difference of pressure from its hydrostatic value; the constant on the right is $\frac{1}{2}c^2$ because at great depths, where $q = c$, this difference is zero. Surface tension creates a pressure difference across the surface which is given by

$$p-p_0=T/R, \tag{3}$$

where p is the pressure in the fluid at the surface, p_0 is the 'atmospheric' pressure, T is the surface tension, and R is the radius of curvature of the surface, counted positive when the centre of curvature lies inside the fluid. The use of p to represent the absolute pressure at the surface as well as its difference from the hydrostatic value is permissible since gravity is neglected as a restoring force. Now when R is positive, d^2y/dx^2 is positive, and so

$$\frac{1}{R}=\frac{d^2y/dx^2}{\{1+(dy/dx)^2\}^{3/2}}. \tag{4}$$

The boundary condition is therefore

$$\frac{T}{\rho}\frac{d^2y/dx^2}{\{1+(dy/dx)^2\}^{3/2}}+\tfrac{1}{2}q^2=\tfrac{1}{2}c^2 \quad \text{on } \psi=0. \tag{5}$$

As the fluid is of infinite depth there are the further conditions

$$\left.\begin{aligned}\frac{\partial\phi}{\partial x}&=\frac{\partial\psi}{\partial y}\to c,\\ \frac{\partial\phi}{\partial y}&=-\frac{\partial\psi}{\partial x}\to 0,\end{aligned}\right\} \quad \text{as } y\to\infty. \tag{6}$$

The problem as formulated is very complicated, but it becomes much simpler on taking new independent variables (ϕ, ψ) and new dependent variables (τ, θ), where $\tau = \log q$ and $(q\cos\theta,\, q\sin\theta)$ are the (x, y)-components of the velocity of the fluid.

On any streamline

$$\frac{1}{R} = \frac{d\theta}{ds}, \tag{7}$$

where ds is an elementary arc of the streamline, so

$$\frac{1}{R} = q\left(\frac{\partial\theta}{\partial\phi}\right)_{\psi=\text{const.}}, \tag{8}$$

and we can write (5) as

$$\frac{T}{\rho} q \frac{\partial\theta}{\partial\phi} + \tfrac{1}{2}q^2 = \tfrac{1}{2}c^2 \quad \text{on } \psi = 0. \tag{9}$$

This can be simplified further if we take $T/\rho c^2$ as unit of length and c as unit of velocity, and this gives

$$e^{\tau}\frac{\partial\theta}{\partial\phi} + \tfrac{1}{2}e^{2\tau} = \tfrac{1}{2} \quad \text{on } \psi = 0. \tag{10}$$

Thus, the boundary condition in the (ϕ, ψ)-plane is

$$\frac{\partial\theta}{\partial\phi} = -\sinh\tau \quad \text{on } \psi = 0. \tag{11}$$

If $w = \phi + i\psi$ and $z = x + iy$,

$$\frac{dw}{dz} = qe^{-i\theta}, \tag{12}$$

and

$$\log\left(\frac{dw}{dz}\right) = \tau - i\theta. \tag{13}$$

These must be regular functions, so we have, equivalent to the original equation of motion (1),

$$\frac{\partial^2\tau}{\partial\phi^2} + \frac{\partial^2\tau}{\partial\psi^2} = 0, \tag{14}$$

and for the same reason we can use the Cauchy–Riemann equations to write (11) as

$$\frac{\partial\tau}{\partial\psi} = -\sinh\tau \quad \text{on } \psi = 0. \tag{15}$$

In the new units $q \to 1$ as $y \to \infty$, so that the conditions (6) become

$$\tau \to 0 \quad \text{and } \theta \to 0 \quad \text{as } \psi \to \infty. \tag{16}$$

3. General solution for τ

To solve this non-linear problem we look for a solution of (14) which satisfies the equation

$$\frac{\partial\tau}{\partial\psi} = -f(\psi)\sinh\tau \tag{17}$$

everywhere in the flow, for some function $f(\psi)$. Such a solution will satisfy the boundary condition (15) provided that

$$f(0) = 1. \tag{18}$$

We can integrate (17) directly:

$$\log\tanh \tfrac{1}{2}\tau = F(\psi) + G(\phi), \tag{19}$$

where

$$\frac{\partial}{\partial\psi} F(\psi) = -f(\psi) \tag{20}$$

and $G(\phi)$ is arbitrary, or

$$\tau = \log\left\{\frac{X(\psi)+Y(\phi)}{X(\psi)-Y(\phi)}\right\}, \tag{21}$$

where

$$X(\psi) = e^{-F(\psi)}, \tag{22}$$

and

$$Y(\phi) = e^{G(\phi)}. \tag{23}$$

Functions $X(\psi)$ and $Y(\phi)$ can now be determined so that Laplace's equation (14) is satisfied. For this to be so we must have

$$2\{(X'(\psi))^2 + (Y'(\phi))^2\} + \{X^2(\psi) - Y^2(\phi)\}\left\{\frac{Y''(\phi)}{Y(\phi)} - \frac{X''(\psi)}{X(\psi)}\right\} = 0. \tag{24}$$

Differentiating this with respect to ϕ and then with respect to ψ gives

$$(X^2(\psi))'\left(\frac{Y''(\phi)}{Y(\phi)}\right)' + \left(\frac{X''(\psi)}{X(\psi)}\right)'(Y^2(\phi))' = 0, \tag{25}$$

or

$$(X^2(\psi))'\Big/\left(\frac{X''(\psi)}{X(\psi)}\right)' + (Y^2(\phi))'\Big/\left(\frac{Y''(\phi)}{Y(\phi)}\right)' = 0. \tag{26}$$

Here each term must be merely a constant, say

$$(X^2(\psi))' = h\left(\frac{X''(\psi)}{X(\psi)}\right)' \tag{27}$$

and

$$(Y^2(\phi))' = -h\left(\frac{Y''(\phi)}{Y(\phi)}\right)'. \tag{28}$$

Integrating, we have

$$X^2(\psi) = h\frac{X''(\psi)}{X(\psi)} + l, \tag{29}$$

$$Y^2(\phi) = -h\frac{Y''(\phi)}{Y(\phi)} + m, \tag{30}$$

where l, m are arbitrary constants, and (24) reduces to

$$2(X'(\psi))^2 - X(\psi)X''(\psi) + m\frac{X''(\psi)}{X(\psi)} + 2(Y'(\phi))^2 - Y(\phi)Y''(\phi) + l\frac{Y''(\phi)}{Y(\phi)} = 0. \tag{31}$$

Hence

$$2(X'(\psi))^2 - X(\psi)X''(\psi) + m\frac{X''(\psi)}{X(\psi)} = n, \tag{32}$$

$$2(Y'(\phi))^2 - Y(\phi)Y''(\phi) + l\frac{Y''(\phi)}{Y(\phi)} = -n, \tag{33}$$

where, again, n is a constant.

From (29)
$$X''(\psi) = \frac{X(\psi)}{h}(X^2(\psi)-l), \tag{34}$$

and from (32)
$$2(X'(\psi))^2 = n + \frac{1}{h}(X^2(\psi)-l)(X^2(\psi)-m). \tag{35}$$

But differentiating (35) gives

$$X''(\psi) = \frac{X(\psi)}{2h}(X^2(\psi)-l+X^2(\psi)-m), \tag{36}$$

and therefore
$$l = m, \tag{37}$$

$$2(X'(\psi))^2 = n + \frac{1}{h}(X^2(\psi)-l)^2, \tag{38}$$

and similarly
$$2(Y'(\phi))^2 = -n - \frac{1}{h}(Y^2(\phi)-l)^2. \tag{39}$$

Thus the general solutions for $X(\psi)$ and $Y(\phi)$ satisfy

$$(X'(\psi))^2 = a_1 + a_2\, X^2(\psi) + a_3\, X^4(\psi), \tag{40}$$

$$(Y'(\phi))^2 = -a_1 - a_2\, Y^2(\phi) - a_3\, Y^4(\phi), \tag{41}$$

where a_1, a_2, a_3 are any constants.

The quadratic case ($a_3 = 0$) has been found to satisfy our requirements for fluid of infinite depth, but inspection of the full quartic has shown that it will give a solution when the fluid depth is finite. It has already been pointed out, however, that the analysis needed is too elaborate to make this solution worthwhile.

4. Details of solution

If we put $a_3 = 0$ in (40) and (41) we have

$$X(\psi) = (a_1/a_2)^{1/2}\sinh(\pm\psi a_2^{1/2} + C), \tag{42}$$

$$Y(\phi) = i(a_1/a_2)^{1/2}\sin(\pm\phi a_2^{1/2} + D), \tag{43}$$

where C, D are constants of integration. Clearly $\{Y(\phi)/X(\psi)\} \to 0$ as $\psi \to \infty$, so that the condition $\tau \to 0$ (16) is satisfied.

From (20) and (22),

$$f(\psi) = \frac{\partial}{\partial\psi}\log X(\psi) \tag{44}$$

$$= \pm k\coth(\pm k\psi + C), \tag{45}$$

writing k for $a_2^{1/2}$, and hence the boundary condition $f(0) = 1$ determines C:

$$C = \pm\{\log(\pm i/A)\}, \tag{46}$$

where
$$A^2 = \frac{k-1}{k+1}. \tag{47}$$

Now, by (21), (42) and (43),

$$\tau = \log\left\{\frac{\sinh(\pm k\psi + C) + \sinh(\pm ik\phi + iD)}{\sinh(\pm k\psi + C) - \sinh(\pm ik\phi + iD)}\right\} \tag{48}$$

$$= \log \frac{\tanh P}{\tanh Q}, \tag{49}$$

where

$$\left.\begin{aligned} P &= \tfrac{1}{2}(\pm k\psi + C \pm ik\phi + iD), \\ Q &= \tfrac{1}{2}(\pm k\psi + C \mp ik\phi - iD). \end{aligned}\right\} \tag{50}$$

Hence,

$$\frac{\partial \tau}{\partial \psi} = \pm \tfrac{1}{2}k\, \frac{\operatorname{sech}^2 P}{\tanh P} \mp \tfrac{1}{2}k\, \frac{\operatorname{sech}^2 Q}{\tanh Q} = \frac{\partial \theta}{\partial \phi}, \tag{51}$$

so that

$$\theta = i \log(\coth P \coth Q) \tag{52}$$

and $\theta \to 0$ as $\psi \to \infty$. Thus

$$\log \frac{dw}{dz} = \log \coth^2 Q, \tag{53}$$

and hence

$$\frac{dw}{dz} = \left(\frac{e^{\mp ikw + C - iD} - 1}{e^{\mp ikw + C - iD} + 1}\right)^2. \tag{54}$$

From (46),

$$e^C = \pm i/A \quad \text{or} \ \mp iA, \tag{55}$$

and, as it only amounts to adding a constant to ϕ, it is permissible to put $D = \frac{1}{2}\pi$. Then we find that both the alternatives (55) lead to

$$\frac{dz}{dw} = \left(\frac{1 \mp Ae^{ikw}}{1 \pm Ae^{ikw}}\right)^2. \tag{56}$$

Taking the alternative sign is equivalent to adding a constant π/k to w (i.e. to ϕ), so the solution is unique:

$$z = w - \frac{4i}{k}\, \frac{1}{1 + Ae^{ikw}} + \frac{4i}{k}. \tag{57}$$

The constant has been chosen to make $z = w$ when $A = 0$. Returning to the original length and velocity units, we get

$$z = \frac{w}{c} - \frac{4i}{k}\, \frac{1}{1 + Ae^{ikw/c}} + \frac{4i}{k}. \tag{58}$$

If w is increased by $2\pi c/k$, the only effect is to increase z by $2\pi/k$, so we have, for the wavelength λ,

$$\begin{aligned} \lambda &= 2\pi/k \\ &= 2\pi \left(\frac{1 - A^2}{1 + A^2}\right) \frac{T}{\rho c^2}. \end{aligned} \tag{59}$$

Hence

$$c = \left(\frac{2\pi T}{\lambda \rho}\right)^{1/2} \left(\frac{1 - A^2}{1 + A^2}\right)^{1/2}, \tag{60}$$

and

$$\frac{z}{\lambda} = \alpha - \frac{2i}{\pi}\, \frac{1}{1 + Ae^{2\pi i \alpha}} + \frac{2i}{\pi}, \tag{61}$$

where
$$\alpha = w/c\lambda, \tag{62}$$
and the range $0 \leqslant \alpha \leqslant 1$ is one wavelength. On the surface $\psi = 0$, $\alpha = (\phi/c\lambda)$ and

$$\left.\begin{aligned} \frac{x}{\lambda} &= \alpha - \frac{2}{\pi}\frac{A\sin 2\pi\alpha}{1+A^2+2A\cos 2\pi\alpha}, \\ \frac{y}{\lambda} &= \frac{2}{\pi} - \frac{2}{\pi}\frac{1+A\cos 2\pi\alpha}{1+A^2+2A\cos 2\pi\alpha}. \end{aligned}\right\} \tag{63}$$

From these relations it is easily seen that, if a is the amplitude of the wave, defined as the vertical height between trough and crest, then

$$\frac{a}{\lambda} = \frac{4A}{\pi(1-A^2)}, \tag{64}$$

or

$$A = \frac{2\lambda}{\pi a}\left\{\left(1+\frac{\pi^2a^2}{4\lambda^2}\right)^{1/2} - 1\right\}. \tag{65}$$

5. Results

The full equation of the surface, in terms of the parameter α, is

$$\frac{z}{\lambda} = \alpha - \frac{2i}{\pi}\left[1+\frac{2\lambda}{\pi a}\left\{\left(1+\frac{\pi^2a^2}{4\lambda^2}\right)^{1/2} - 1\right\}e^{2\pi i\alpha}\right]^{-1} + \frac{2i}{\pi}. \tag{66}$$

Preliminary calculations showed that for large enough values of a/λ, the wave surface crosses itself, and this suggested that the wave of greatest height would occur at the value of a/λ for which the surface was tangent to itself.

For this to be possible we must have $x = 0$ for $\alpha \neq 0$; i.e.

$$1+A^2+2A\cos 2\pi\alpha = \frac{4A\sin 2\pi\alpha}{2\pi\alpha}, \tag{67}$$

or

$$A^2 - 2Af(2\pi\alpha) + 1 = 0, \tag{68}$$

where

$$f(t) = \frac{2\sin t}{t} - \cos t, \tag{69}$$

and we want the least value of a/λ for which this condition is satisfied; i.e.

$$A = f(t_0) - \{(f(t_0))^2 - 1\}^{1/2}, \tag{70}$$

where $f(t_0)$ is the minimum value of $f(t)$. This gives for the wave of greatest height

$$a/\lambda = 0{\cdot}730. \tag{71}$$

By contrast, the gravity wave of greatest height has $a/\lambda = 0{\cdot}142$ (Michell 1893).

An interesting point to notice is that the surface $\psi = 0$ for waves with amplitude corresponding to $A = A_1$ is the streamline $\psi = \text{const.}$, given by

$$A_2 e^{-2\pi\psi/c\lambda} = A_1 \tag{72}$$

for waves with amplitude corresponding to $A = A_2$ $(A_2 > A_1)$. This fact simplifies the computation because the streamlines for the wave of greatest height, shown in figure 1, are themselves surface shapes for the

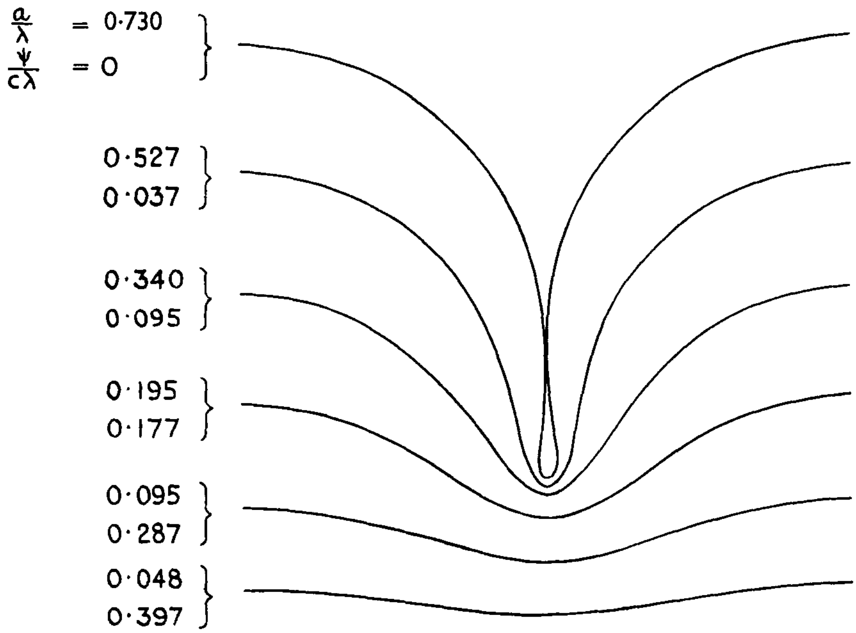

Figure 1. Streamlines for the wave of greatest height. Each line is itself the surface shape for the stated value of a/λ, and if any particular one is taken as surface those below it are still streamlines.

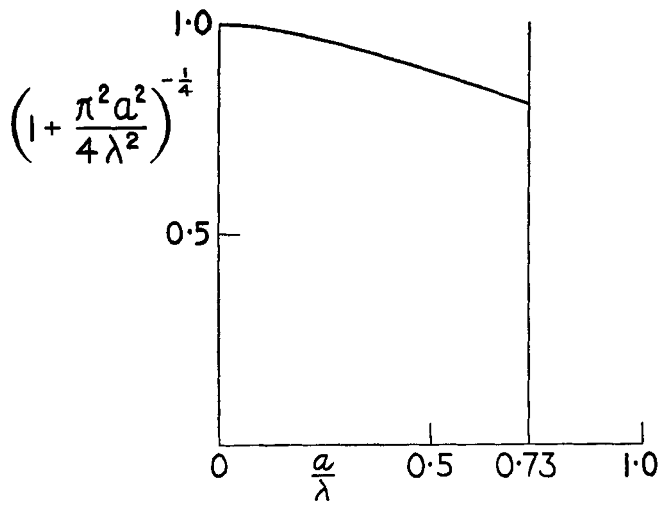

Figure 2. Ratio of wave velocity for amplitude a to wave velocity for zero amplitude for waves of length λ.

stated values of (a/λ), and if any particular line is taken as surface, those below it are still streamlines. The almost circular shape of the crests in figure 1 is to be expected in what the streamlines indicate to be a region of slow flow, since here surface tension is the dominating force.

Finally we have, from (60) and (64),

$$c = \left(\frac{2\pi}{\lambda}\frac{T}{\rho}\right)^{1/2}\left(1+\frac{\pi^2 a^2}{4\lambda^2}\right)^{-1/4}. \tag{73}$$

The function $(1+\pi^2a^2/4\lambda^2)^{-1/4}$, which shows how the wave velocity falls away from the value given by linear theory as the amplitude increases, is drawn in figure 2.

The author is greatly indebted to Professor M. J. Lighthill for much valuable help with the research and preparation of the paper, and to the Department of Scientific and Industrial Research for a maintenance grant.

References

Lamb, H. 1932 *Hydrodynamics*, 6th Ed. Cambridge University Press.

Michell, J. H. 1893 *Phil. Mag.* (5), **36**, 430.

Stokes, G. G. 1847 *Trans. Camb. Phil. Soc.* **8**, 441.

Stokes, G. G. 1880 *Mathematical and Physical Papers*, Vol. 1. Cambridge University Press.